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Overall electromechanical properties of a binary composite with 622 symmetry constituents. Antiplane shear piezoelectric state

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Abstract

A binary composite is studied here, where the electroelastic properties of the constituent materials belong to the crystal class 622. A square arrangement of long continuous circular cylinders, the fiber phase, embedded in a homogeneous medium is consider here. The composite is in a state of antiplane shear piezoelectricity, that is, a coupled state of out-of-plane mechanical displacement and in-plane electric field, which is characterized by three electroelastic parameters: longitudinal shear modulus, shear stress piezoelectric coefficient and transverse dielectric constant. Our interest here lies in the determination of its effective properties. They are derived by means of the method of two spatial scales. Closed-form expressions are obtained for them. Only one of the four local (or canonical) problems that arise is needed. Two properties are thus found. The Milgrom–Shtrikman compatibility relation is used to fix the remaining one. The local problem is solved using potential methods of a complex variable. The solution involves doubly periodic Weierstrass elliptic and related functions. The final formulae for the overall properties show explicitly the dependence on (i) the properties of the phases, (ii) the radius of the cylindrical fiber and (iii) the lattice sums associated with the square array. The shear modulus is shown to depend explicitly not only on the rigidity of the phases but also on their piezoelectric and dielectric coefficients. Some natural organic substances have the symmetry 622 like collagen. Recently Silva et al. measured its electroelastic properties. Their data is used to show some numerical results of the derived formulae as a function of the fiber volumetric fraction.

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1. Introduction

Composites, whether natural or man-made, span a wide variety of geometries (particulates, fibers, laminates, etc.). Its composition is guided, and even tailored, by the application at hand (Hull and Clyne, 2000; Kelly and Sweeny, 2000). One example widely studied among many others considers long parallel continuous fibers of one material embedded in a binding material. When the lateral diameter of a fiber is small compared with the size of the whole, an important matter arises which belongs to the subject of micromechanics. Is it possible to predict the overall properties in the bulk once the constituent properties, volume fractions and architecture are known? Quite a number of analytical and numerical techniques have been used to answer this question. References to these can be found in the textbooks mentioned in Rodríguez-Ramos et al. (2001) and Guinovart-Díaz et al. (2001). In particular, many applications to other fields using the asymptotic homogenization method are referred to in these papers. In general the application of the latter method to a periodic medium leads necessarily to the solution of several so-called local (or canonical) problems which take place on a periodic unit cell. These are usually solved using numerical methods (e.g., Galka et al., 1996; Pastor, 1997; Andrianov et al., 2002). However, for a two-phase composite exact closed-form effective expressions have been found for certain anisotropic materials and arrays of circular cylinders: square (Rodríguez-Ramos et al., 2001; Bravo-Castillero et al., 2001; Valdiviezo-Mijangos et al., 2002b) and hexagonal (Guinovart-Díaz et al., 2001; Sabina et al., 2001; Valdiviezo-Mijangos et al., 2002a) arrays and materials with 6 mm and cubic symmetry. Recently, Silva et al. (2001) have measured three electroelastic properties of two films: anionic collagen and a composite collagen-hydroxyapatite (HA). The purpose of their research is to look for new biomaterials having the potential of guided osteogenesis. It is interesting to note that the three measured properties correspond to the 622 crystal class, which together with class 6 is common in some natural organic substances (Fukada, 1984; Ikeda, 1990). Both collagen and HA are natural constituents of bone and have good qualities for use in medical applications (Thompson and Hench, 2000). This combination is also under study for bone graft devices (Mythili et al., 2000) in which case the microstructure is rather complex, that of two interpenetrating phases. A simple model of a fiber-reinforced composite with natural materials showing this kind of piezoelectricity can be theoretically studied using the asymptotic homogenization method to calculate overall properties. The data on the electroelastic properties of solids of classes 6 and 622 is, however, not complete. Due to the cylindrical symmetry of the array, it turns out that each plane perpendicular to the axis of symmetry (material and geometric) behaves in the same way and only requires the knowledge of the properties measured by Silva et al. (2001). Thus, it is proposed to address the problem of the prediction of the overall properties for a square array of a binary composite with constituents properties of the 622 symmetry.

The paper is organized as follows. Section 2 introduces the constitutive relations, the equilibrium equations and the interface conditions between two materials of class 622. In the next section, the method of homogenization is briefly mentioned. The calculation of the overall properties are dependent on the solution of certain local problems. One such problem, the $13L$ one, is stated and its relation to some of the properties is shown. Section 4 uses potential theory and the properties of doubly periodic elliptic functions to construct the solution of the local problem stated in Section 3. The closed-form formulae for the overall properties are derived in Section 4. A numerical example is shown in Section 5. Finally, Section 6 has some concluding remarks. The paper ends with two appendices A, B which collect some formulae.

Use is made of general formulae presented in Bravo-Castillero et al. (2001); references to equations and sections from that source are given the prefix **II**.

2. Statement of the problem

A two-phase periodic composite is considered here which consists of a square array of identical parallel circular cylinders embedded in a homogeneous medium. The cylinders are infinitely long (Fig. 1). The material electroelastic properties of each phase belong to the class 622, where the axes of material and geometric symmetry are parallel to the 3 direction. The governing electroelastic equations for this kind of materials are the Navier equations of linear elasticity and Maxwell's quasistatic equations for the mechanical displacement $u = (u, v, w)$ and electric field $\mathbf{E} = (E_1, E_2, E_3)$. They become coupled equations for \mathbf{u} and \mathbf{E} through the constitutive relations of the medium. In a two-dimensional situation, like in the considered geometry here, it is well-known that the equations of elasticity (with no piezoelectricity present) for, say, two isotropic solids uncouple into two independent systems under suitable boundary conditions. Namely, the familiar plane- and antiplane-strain deformation states. The in-plane displacements components $u(x_1, x_2)$, $v(x_1, x_2)$ only appear in the former state. In the later one the remaining out-of-plane displacement $w(x_1, x_2)$ is present (see, e.g., Nemat-Nasser and Hori, 1999, p. 82). A similar situation arises when there is piezoelectric coupling, i.e., for the full electroelastic equations Benveniste (1995) has shown that, under certain loading conditions at the (cylindrical) external boundary of solids of class 2, the electroelastic equations uncouple also into two separate problems. Although it is not mentioned explicitly there, the same uncoupling occurs for solids of the class 622 considered here. One of them, involves u , v , E_3 , i.e., it is a state of in-plane mechanical deformation and out-of-plane electric field. The other state, which is of particular interest here, is characterized by an out-of-plane mechanical displacement w and an in-plane electric field E_1 , E_2 . The main aim of this paper is the determination of effective properties using the homogenization method as in Bravo-Castillero et al. (2001). In that paper the same cylindrical geometry is considered except that the phases are solids of the class 6mm. The above-mentioned uncoupling occurs, but it turns out that the electromechanical variables u , v , E_3 satisfy the same equations as the class 622. It is not so for the other electromechanical state. Thus it is only necessary to solve for the remaining one relating w , E_1 , E_2 . In this case the relevant constitutive relations are

$$\sigma_{23} = 2p\epsilon_{23} - s'E_1, \quad \sigma_{13} = 2p\epsilon_{13} + s'E_2, \quad D_1 = 2s'\epsilon_{23} + tE_1, \quad D_2 = -2s'\epsilon_{13} + tE_2, \quad (2.1)$$

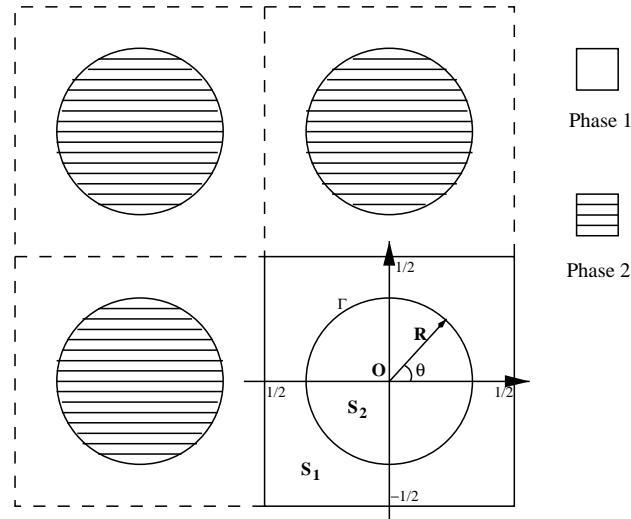


Fig. 1. Binary composite: cross-section of a square array of circular cylinders of radius R . At the lower right bottom without the azure lines, the coordinate system used is shown on the periodic unit cell $S = S_1 \cup S_2$; the common interface being denoted Γ .

where σ_{13} , σ_{23} are stress components; ϵ_{13} , ϵ_{23} those of strain; D_1 , D_2 in-plane electric displacement components; only three material properties appear here, namely, the longitudinal shear modulus p , the transverse permittivity constant t and the shear stress piezoelectric coefficient s' . In fact, this material has only one piezoelectric modulus (Nye, 1998).

The equilibrium equations in the composite are

$$\sigma_{13,1} + \sigma_{23,2} = f, \quad D_{1,1} + D_{2,2} = 0, \quad (2.2)$$

where the comma notation is understood to denote differentiation with respect to x_i , i.e., $D_{1,1} \equiv \partial D_1 / \partial x_1$ and f is the body force (Ikeda, 1990).

Let p_1 , s'_1 , t_1 be the material properties, and φ_1 the electric potential in the matrix. Similarly a subindex 2 is used for variables associated with the fiber. Note the differential relations

$$2\epsilon_{13} = w_{,1}, \quad 2\epsilon_{23} = w_{,2}, \quad E_1 = -\varphi_{,1}, \quad E_2 = -\varphi_{,2}. \quad (2.3)$$

The two phases are assumed to be in perfect contact along the interface of each cylinder which is denoted by Γ (Fig. 1) and satisfy the conditions of continuity of displacement, potential, traction and normal component of electric displacement. Thus

$$\|w\| = 0 \quad \text{on } \Gamma, \quad \|\varphi\| = 0 \quad \text{on } \Gamma, \quad \|\sigma_{13}n_1 + \sigma_{23}n_2\| = 0 \quad \text{on } \Gamma, \quad \|D_1n_1 + D_2n_2\| = 0 \quad \text{on } \Gamma, \quad (2.4)$$

where $\mathbf{n} = (n_1, n_2)$ is the unit normal vector to Γ , and the double bar notation is used to denote the jump of the relevant function across Γ taken from the matrix to the fiber.

3. The method of solution

Now let l be the distance between the centers of two neighbouring cylinders and L the diameter of the composite. Then, when $\epsilon = l/L$ is a very small number, it is possible to distinguish two spatial scales, one is \mathbf{x} , the slow variable, and the other is $\mathbf{y} = \mathbf{x}/\epsilon$, the fast variable. The boundary value problem (2.1)–(2.4) in the composite with Benveniste's boundary conditions can be solved asymptotically posing the ansatz

$$w(\mathbf{x}) = w_0(\mathbf{x}, \mathbf{y}) + \epsilon w_1(\mathbf{x}, \mathbf{y}) + \mathcal{O}(\epsilon^2), \quad \varphi(\mathbf{x}) = \varphi_0(\mathbf{x}, \mathbf{y}) + \epsilon \varphi_1(\mathbf{x}, \mathbf{y}) + \mathcal{O}(\epsilon^2) \quad (3.1)$$

in (2.1)–(2.4) using the method of two scales. The functions w_0 , φ_0 , w_1 , φ_1 are found to satisfy certain differential equations related to the original system in a unit cell (see Fig. 1) with periodic conditions. It is a well-known derivation whose details can be found elsewhere (e.g., Parton and Kudryavtsev, 1993) and is omitted. Of a greater interest here are the so-called local (or canonical) problems associated with the correction terms w_1, φ_1 to the mean variations w_0, φ_0 since they appear in the formulae for the effective properties. There are four of such problems, which are referred as $_{13}L$, $_{23}L$, ${}_1L$ and ${}_2L$. A preindex is used to distinguish similar constants and functions such as displacements and potentials, which appear below. Due to the linearity of the equations (2.1)–(2.4), the corrections terms w_1 , φ_1 can be obtained as a linear combination of such displacements and potentials. This, however, will not be done here, since the main objective of this paper is the characterization of the effective properties p , s' and t . Explicit relationships for all of them are collected in Appendix A. There are several alternatives for each property: two for p and t and four for s' . As a start, one of them is chosen. It requires the solution of a local problem, say, $_{13}L$. This means that is necessary to consider, (A.1a,c), viz.,

$$p = p_v + \langle pM_{,1} - s'N_{,2} \rangle, \quad s' = s'_v + \langle s'M_{,1} + tN_{,2} \rangle, \quad (3.2)$$

where the preindex 13 is now dropped since no further confusion can arise in the rest of the paper. The displacement $M^{(\Upsilon)}$ and potential $N^{(\Upsilon)}$ ($\Upsilon = 1, 2$) are the solution of the corresponding two indices (13) local problem, taken from equations (II, 2.4–5), viz.,

$$\begin{aligned} \Delta M^{(r)} &= 0 \quad \text{in } S_r, \quad \Delta N^{(r)} = 0 \quad \text{in } S_r, \quad \|M^{(r)}\| = 0 \quad \text{on } \Gamma, \quad \|N^{(r)}\| = 0 \quad \text{on } \Gamma, \\ \|(p_r M_{,1}^{(r)} - s'_r N_{,2}^{(r)}) n_1 + (p_r M_{,2}^{(r)} + s'_r N_{,1}^{(r)}) n_2\| &= -\|p_r\| n_1 \quad \text{on } \Gamma, \\ \|(s'_r M_{,2}^{(r)} - t_r N_{,1}^{(r)}) n_1 - (s'_r M_{,1}^{(r)} + t_r N_{,2}^{(r)}) n_2\| &= \|s'_r\| n_2 \quad \text{on } \Gamma, \quad \langle M \rangle = 0, \quad \langle N \rangle = 0, \end{aligned} \quad (3.3a-h)$$

where Δ is the two-dimensional Laplacian. Thus $M^{(r)}$ and $N^{(r)}$ are sought such that they are doubly periodic harmonic functions of the complex variable $z = y_1 + iy_2$ in the square unit cell $S (= S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$) of periods $\omega_1 = 1$, $\omega_2 = i$. A limiting case is noted here which is useful below. When the piezoelectric coefficients s'_1 , s'_2 vanish, there is no electroelastic coupling. Hence the equations (3.3a-h) uncouple in two independent sets. Those for $M^{(r)}$ correspond to the antiplane elastic problem ${}_{13}L$ (II, Section 3.2). The remaining equations for $N^{(r)}$ are homogeneous implying either a null potential or the existence of resonances (McPhedran and McKenzie, 1980) for the dielectric problem. However, the non-resonant contribution to the correction term φ_1 in (3.1) comes from the ${}_1L$ or ${}_2L$ local problem, i.e., the effective permittivity follows from Eqs. (B.2g,h), see also Table 1.

Eqs. (3.2) are easily transformed applying Green's theorem to the area integrals. The doubly periodic boundary conditions on S and the continuity of displacement and potential on Γ leads to

$$p = p_v - \|p_r\| \int_{\Gamma} M^{(2)} dy_2 - \|s'_r\| \int_{\Gamma} N^{(2)} dy_1, \quad s' = s'_v - \|s'_r\| \int_{\Gamma} M^{(2)} dy_2 + \|t_r\| \int_{\Gamma} N^{(2)} dy_1. \quad (3.4)$$

The same procedure is applied to all of (A.1a-h). The results are collected in Appendix B.

4. Solution of the local problem ${}_{13}L$

The methods of potential theory are used to solve (3.3a-h). Doubly periodic harmonic functions are to be found in terms of the following expansions of harmonic functions:

$$\begin{aligned} M^{(1)}(z) &= \operatorname{Re} \left\{ -\pi a_1 z + \sum_{k=1}^{\infty} a_k \frac{\zeta^{(k-1)}(z)}{(k-1)!} \right\}, \\ N^{(1)}(z) &= \operatorname{Im} \left\{ \pi b_1 z + \sum_{k=1}^{\infty} b_k \frac{\zeta^{(k-1)}(z)}{(k-1)!} \right\}, \\ M^{(2)}(z) &= \operatorname{Re} \left\{ \sum_{k=1}^{\infty} c_k z^k \right\}, \\ N^{(2)}(z) &= \operatorname{Im} \left\{ \sum_{k=1}^{\infty} d_k z^k \right\}, \end{aligned} \quad (4.1a-d)$$

where a_k , b_k , c_k , d_k , are real undetermined coefficients and $\zeta(z)$ is the quasi-periodic Weierstrass Zeta function of periods ω_1 and ω_2 , $\zeta^{(k)}(z)$ is its k th derivative, which are doubly periodic of periods ω_1 , ω_2 . The

Table 1
Local problems, functions and jump conditions

Problem	${}_{13}L$	${}_{23}L$	${}_1L$	${}_2L$
Displacement	${}_{13}M$	${}_{23}M$	${}_1P$	${}_2P$
Potential	${}_{13}N$	${}_{23}N$	${}_1Q$	${}_2Q$
RHS (3.3e)	$-\ p_r\ n_1$	$-\ p_r\ n_2$	$-\ s'_r\ n_2$	$\ s'_r\ n_1$
RHS (3.3f)	$\ s'_r\ n_2$	$-\ s'_r\ n_1$	$\ t_r\ n_1$	$\ t_r\ n_2$

superindex o next to the sum symbol means that k runs only over odd integers so that each term in (4.1a–d) has the same antisymmetry property as $M^{(Y)}$ and $N^{(Y)}$, namely, $M^{(Y)}(-z) = -M^{(Y)}(z)$, $N^{(Y)}(-z) = -N^{(Y)}(z)$. In addition, $M^{(Y)}(N^{(Y)})$ is an even (odd) function of θ , where $z = re^{i\theta}$; this follows, say, by examination of the right-hand side of (3.3e,f). The first term in each expansion (4.1a,b) arises due to the quasi-periodicity property of the $\zeta(z)$, which is

$$\zeta(z + \omega_\alpha) - \zeta(z) = \delta_\alpha, \quad \alpha = 1, 2, \quad (4.2)$$

where $\omega_1 = 1$, $\omega_2 = i$, $\delta_1 = \pi$, $\delta_2 = -i\pi$.

The line integrals in (3.4) and the assumed expansions (4.1c,d) produce a very simple result as a consequence of the orthogonality of the trigonometric functions, namely,

$$p = p_v - (\|p_Y\|c_1 - \|s'_Y\|d_1)\pi R^2, \quad s' = s'_v - (\|s'_Y\|c_1 + \|t_Y\|d_1)\pi R^2. \quad (4.3)$$

Once p and s' are found it remains to seek t to characterize all the sought overall properties.

The method developed here is a possible way, which requires the solution of another local problem (see, (A.1a–h)). Another one, simpler, consists in the application of the compatibility condition of Milgrom and Shtrikman (1989).

The Laurent expansion of (4.1a,b) about the origin is easily found to be

$$\begin{aligned} M^{(1)}(z) &= \operatorname{Re} \left\{ \sum_{l=1}^{\infty} a_l z^{-l} - \sum_{k=1}^{\infty} a_k \sum_{l=1}^{\infty} k \eta_{kl} z^l \right\}, \\ N^{(1)}(z) &= \operatorname{Im} \left\{ \sum_{l=1}^{\infty} b_l z^{-l} - \sum_{k=1}^{\infty} b_k \sum_{l=1}^{\infty} k \eta'_{kl} z^l \right\}, \end{aligned} \quad (4.4)$$

where

$$\eta_{11} = -\eta'_{11} = \pi, \quad \eta_{kl} = \eta'_{kl} = \frac{(k+l-1)!}{k!l!} S_{k+l} \quad \text{for } k, l \neq 1, \quad (4.5)$$

the definitions of η_{kl} , η'_{kl} here are different from those used in Rodríguez-Ramos et al. (2001); the lattice sums S_k are defined by

$$S_k = \sum'_{m,n} (m\omega_1 + n\omega_2)^{-k} \quad k \geq 3, \quad (4.6)$$

where the prime means that the summation over all the integers does not include the term $m = n = 0$. The series are absolutely and uniformly convergent (Markushevich, 1970, p. 335). The interface conditions (3.3c–f) are now used to establish relationships among the coefficients in (4.1a), (4.2)–(4.4). These are

$$\begin{aligned} R^l c_l &= R^{-l} a_l - \sum_{k=1}^{\infty} k \eta_{kl} R^l a_k, \\ R^l d_l &= -R^{-l} b_l - \sum_{k=1}^{\infty} k \eta'_{kl} R^l b_k, \\ -\|p_Y\| R \delta_{1l} &= -(p_1 + p_2) R^{-l} a_l - \|p_Y\| \sum_{k=1}^{\infty} k \eta_{kl} R^l a_k + \|s'_Y\| \left(R^{-l} b_l + \sum_{k=1}^{\infty} k \eta'_{kl} R^l b_k \right), \\ -\|s'_Y\| R \delta_{1l} &= \|s'_Y\| \left(R^{-l} a_l - \sum_{k=1}^{\infty} k \eta_{kl} R^l a_k \right) + (t_1 + t_2) R^{-l} b_l - \|t_Y\| \sum_{k=1}^{\infty} k \eta'_{kl} R^l b_k, \end{aligned} \quad (4.7a-d)$$

for $l = 1, 3, 5, \dots$. Once the multipole coefficients, the a 's and b 's, are found from (4.7c,d) the remaining ones are obtained from (4.7a,b).

A closer look at the system (4.7a–d) for $l = 1$ gives the relations

$$\begin{aligned} Rc_1 &= R^{-1}a_1 - \sum_{k=1}^{\infty} k\eta_{k1}Ra_k, \\ Rd_1 &= -R^{-1}b_1 - \sum_{k=1}^{\infty} k\eta'_{k1}Rb_k, \\ -\|p_T\|R &= -(p_1 + p_2)R^{-1}a_1 - \|p_T\| \sum_{k=1}^{\infty} k\eta_{k1}Ra_k + \|s'_T\| \left(R^{-1}b_1 + \sum_{k=1}^{\infty} k\eta'_{k1}Rb_k \right), \\ -\|s'_T\|R &= \|s'_T\| \left(R^{-1}a_1 - \sum_{k=1}^{\infty} k\eta_{k1}Ra_k \right) + (t_1 + t_2)R^{-1}b_1 - \|t_T\| \sum_{k=1}^{\infty} k\eta'_{k1}Rb_k. \end{aligned} \quad (4.8)$$

The two different summations in (4.8) can be eliminated to yield simpler connections between c_1 , d_1 and a_1 , b_1 . These are

$$\|p_T\|c_1 - \|s'_T\|d_1 = 2p_1R^{-2}a_1 - \|p_T\|, \quad \|s'_T\|c_1 + \|t_T\|d_1 = -2t_1R^{-2}b_1 - \|s_T\|. \quad (4.9)$$

With these results, (4.3) become

$$p = p_1(1 - 2\pi a_1), \quad s' = s'_1 + 2\pi t_1 b_1 \quad (4.10a,b)$$

in which only the residue of $M^{(1)}(N^{(1)})$ contributes toward $p(s')$. The expression for p in (4.10a) is the same as in the pure elastic case (II, 3.11). The coefficient a_1 , however, is different in both cases. Thus, expressions for a_1 , b_1 are now sought.

It is convenient to introduce new scaled variables (McPhedran and McKenzie, 1980)

$$a'_l = \sqrt{l}R^{-l}a_l, \quad b'_l = \sqrt{l}R^{-l}b_l, \quad c'_l = \sqrt{l}R^l c_l, \quad d'_l = \sqrt{l}R^l d_l \quad (4.11)$$

in (4.7a–d), so that they become

$$\begin{aligned} (I - W)D_1 &= D_3, \quad (I + W')D_2 = -D_4, \quad \varphi_{11}^{(1)}D_1 + \varphi_{12}^{(1)}D_2 + \varphi_{11}^{(2)}WD_1 + \varphi_{12}^{(2)}WD_2 = \overline{U}_1, \\ \varphi_{21}^{(1)}D_1 + \varphi_{22}^{(1)}D_2 + \varphi_{21}^{(2)}W'D_1 + \varphi_{22}^{(2)}W'D_2 &= \overline{U}_2, \end{aligned} \quad (4.12a-d)$$

where I is the identity matrix, the components of the matrices W and W' are for $k = l = 1$

$$w_{11} = \pi R^2 = -w'_{11}, \quad (4.13)$$

and otherwise

$$w_{kl} = w'_{kl} = \frac{(k+l-1)!}{(k-1)!(l-1)!} \frac{R^{k+l}}{\sqrt{k}\sqrt{l}} S_{k+l}, \quad (4.14)$$

the matrices W , W' are real, symmetric and bounded; the introduction of W' although identical to W except for w'_{11} is quite useful;

$$\begin{aligned} D_1 &= (a'_1 \quad a'_3 \quad a'_5 \quad \dots)^T, \quad D_2 = (b'_1 \quad b'_3 \quad b'_5 \quad \dots)^T, \\ D_3 &= (c'_1 \quad c'_3 \quad c'_5 \quad \dots)^T, \quad D_4 = (d'_1 \quad d'_3 \quad d'_5 \quad \dots)^T; \end{aligned} \quad (4.15)$$

all the components of the vectors \overline{U}_1 , \overline{U}_2 are zero except the first one, which are $R\chi_p$, $R\chi'_t$ respectively, where

$$\chi_p = \frac{\|p_T\|}{p_1 + p_2}, \quad \chi'_t = \frac{\|s'_T\|}{t_1 + t_2}, \quad \chi'_p = \frac{\|s'_T\|}{p_1 + p_2}, \quad \chi_t = \frac{\|t_T\|}{t_1 + t_2}; \quad (4.16)$$

the last two definitions are conveniently introduced here; it remains to define the 2×2 non-symmetric matrices $\Phi^{(1)}, \Phi^{(2)}$ whose components appear in (4.12c,d). They are

$$\Phi^{(1)} = \begin{bmatrix} 1 & -\chi'_p \\ -\chi'_t & -1 \end{bmatrix}, \quad \Phi^{(2)} = \begin{bmatrix} \chi_p & -\chi'_p \\ \chi'_t & \chi_t \end{bmatrix}. \quad (4.17)$$

Therefore Eqs. (4.12c,d) are transformed into

$$\varphi_{11}D_1 + \varphi_{12}D_2 + WD_1 = U_1, \quad \varphi_{21}D_1 + \varphi_{22}D_2 + W'D_2 = 0, \quad (4.18a,b)$$

where only the first component of U_1 is non-zero, it is equal to R and Φ^{-1} is the 2×2 non-symmetric matrix

$$\Phi^{-1} = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = [\Phi^{(2)}]^{-1} \Phi^{(1)} = \frac{1}{\Delta} \begin{bmatrix} \chi_t - \chi'_p \chi'_t & -\chi'_p(1 + \chi_t) \\ -\chi'_t(1 + \chi_p) & -\chi_p + \chi'_p \chi'_t \end{bmatrix}, \quad \Delta = \chi_p \chi_t + \chi'_p \chi'_t. \quad (4.19)$$

When there is no coupling $s'_1 = s'_2 = 0$ or $\chi'_p = \chi'_t = 0$, then

$$\Phi^{-1} = \text{diag}(\chi_p^{-1}, -\chi_t^{-1}). \quad (4.20)$$

The system (4.18a,b) uncouples into

$$(\chi_p^{-1}I + W)D_1 = U_1, \quad (-\chi_t^{-1}I + W')D_2 = 0. \quad (4.21a, b)$$

The first system (4.21a) agrees with the result, Eq. (7), of McPhedran and McKenzie (1980) derived in the context of dielectrics using Rayleigh's method (1892). The matrix W plays an important role there as it is the case here with W and W' . It must be mentioned that in the analysis carried out here the lattice sum S_2 does not appear at all unlike in McPhedran and McKenzie (1980). Here the contribution of π in w_{11} and w'_{11} is due to the doubly periodicity of the functions in (4.1a,b). Although using a different notation, (4.21a) is also given as Eq. (3.9) of Rodriguez-Ramos et al. (2001) for the elasticity problem. The other system (4.21b) is homogeneous, so $D_2 = 0$ if χ_t^{-1} is not an eigenvalue of W' . Otherwise there appears to be resonant solutions. The analysis of these is beyond the scope of the present paper.

The system of equations (4.18a,b) has a useful particular structure which is due to the periodicity of the square array. The components w_{kl} of W vanish whenever $k + l$ is not a multiple of four. The system (4.18a,b) can be reorganized in three systems considering (i) the first equation in each system (4.18a,b), (ii) the set of equation with Greek index $\alpha = 3, 7, 11, \dots$ and (iii) the set with Latin index $i = 5, 9, 13, \dots$. From this point onwards, the use of Greek and Latin as subindices runs as defined. The system can be written as follows

$$(\Phi^{-1} + \Psi)A_1 + \mathcal{V}^T D^{(\alpha)} = B, \quad F^{-1}D^{(\alpha)} + G\mathcal{V} + \mathcal{W}D^{(i)} = 0, \quad F^{-1}D^{(i)} + \mathcal{W}^T D^{(\alpha)} = 0, \quad (4.22a-c)$$

where the 2×1 vectors A_J , B and 2×2 matrices Ψ , I_2 are

$$A_J = (a'_J \quad b'_J)^T \quad \text{for } J = 1, 3, 5, \dots, \quad B = (R \quad 0)^T, \quad \Psi = \begin{pmatrix} w_{11} & 0 \\ 0 & w'_{11} \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.23)$$

the vectors partitioned in blocks $2 \times 1 D^{(\alpha)}$ and $D^{(i)}$; \mathcal{V} in blocks 2×2 are:

$$D^{(\alpha)} = (A_3 \quad A_7 \quad A_{11} \quad \dots)^T, \quad D^{(i)} = (A_5 \quad A_9 \quad A_{13} \quad \dots)^T, \quad \mathcal{V}^T = (w_{13}I_2 \quad w_{17}I_2 \quad w_{1,11}I_2 \quad \dots); \quad (4.24)$$

the square matrices partitioned in 2×2 blocks \mathcal{W} , F and 1×2 blocks G are

$$\mathcal{W} = \begin{bmatrix} w_{35}I_2 & w_{39}I_2 & \dots \\ w_{75}I_2 & w_{79}I_2 & \dots \\ \vdots & \vdots & \end{bmatrix}, \quad F = \begin{bmatrix} \Phi & 0 & \dots \\ 0 & \Phi & \dots \\ \vdots & \vdots & \end{bmatrix}, \quad G = \begin{pmatrix} A_1 & 0 & \dots \\ 0 & A_1 & \dots \\ \vdots & \vdots & \end{pmatrix}. \quad (4.25)$$

The system of Eqs. (4.22a–c) in the unknowns A_1 , $D^{(\alpha)}$ and $D^{(i)}$ is such that A_1 can be found explicitly in the following way. The $D^{(i)}$ found in (4.22c) is substituted into (4.22b). Here the system of equations is solved for $D^{(\alpha)}$ and then put in (4.22a), which leaves an equation for A_1 . This method was used by Pobedrya (1984), who showed several examples of alike systems which can be solved for the lowest order coefficient in the Laurent expansion. Thus

$$A_1 = [I_2 + \Phi\Psi - \Phi^2\mathcal{V}^T\mathcal{M}^{-1}\mathcal{V}]^{-1}\Phi B, \quad (4.26)$$

where

$$\mathcal{M} = I - F^2\mathcal{W}\mathcal{W}^T \quad (4.27)$$

and its components are 2×2 matrices

$$m_{\alpha\beta} = I_2 - \Phi^2 \sum_{i=5}^{\infty} w_{\alpha i} w_{i\beta}. \quad (4.28)$$

When $s'_1 = s'_2 = 0$ (or $\chi'_p = \chi'_t = 0$), the components of A_1 in (4.26) become $b_1 = 0$ and

$$a_1 = \chi_p R^2 \left[1 + \chi_p w_{11} - \chi_p^2 \mathcal{V}^T \mathcal{M}^{-1} \mathcal{V} \right]^{-1}, \quad (4.29)$$

where the components of \mathcal{M} and \mathcal{V} are given now without block partitions,

$$m_{\alpha\beta} = \delta_{\alpha\beta} - \chi_p^2 \sum_{i=5}^{\infty} w_{\alpha i} w_{i\beta}, \quad v_{\alpha} = w_{1\alpha}. \quad (4.30)$$

Expression (4.29) agrees with the corresponding one in (II, 3.13) derived for the antiplane elastic case, although the notation and some definitions are different. Note that the matrix Φ in (4.26) plays a similar role as the scalar χ_p in (4.29). Both contain information about material properties of the phases. It is also interesting to see that $w_{11} = \pi R^2 = V_2$, the fiber volume fraction. Now the effective properties p , s' can be found using (4.10a,b). They depend explicitly on (i) the properties of the phases, (ii) the radius of the cylindrical fiber and (iii) the lattice sums associated with the square array. In Appendix B formulae are given for the effective properties in terms of the a_1 and b_1 coefficients which correspond to all the local problems.

The other three local problems are very much alike (3.3a–h). With reference to Table 1, the associated equations to $_{23}L$ can be obtained from (3.3a–h). One must substitute the elements of the second column (those of the $_{23}L$ problem) instead of the elements of the first one (corresponding to $_{13}L$); and so on for the remaining $_{1}L$ and $_{2}L$.

Because of the linearity of the canonical equations and the same angular dependence on the right-hand side of (3.3e,f), it turns out that the solution of the two local problems $_{13}L$ ($_{23}L$) and $_{2}L$ ($_{1}L$) are closely related. By keeping track of the coefficients in the right-hand side of (3.3e,f), one can get the solutions of the other related problem, is therefore only necessary to solve **two** local problems. Or only **one**, if use is made of the universal relation among the three properties.

5. Numerical example

The constituent properties for the calculations that follow were taken from Silva et al. (2001) in their study of piezoelectric properties of films of biomaterials of the class 622. Here it is assumed that the measured properties also refer to the bulk properties. After some minor calculations and correction of units, the data used are: for collagen, which is taken as the matrix medium, $p_1 = 1.4$ GPa, $t_1/\epsilon_0 = 2.825$ ($\epsilon_0 = 8.854 \times 10^{-12}$ C 2 /N m 2 is the permittivity of free space), $d_1 = 0.062$ pC/N; the fiber material is a collagen-hydroxyapatite (HA) composite, whose properties are $p_2 = 2.697$ GPa, $t_2/\epsilon_0 = 2.509$,

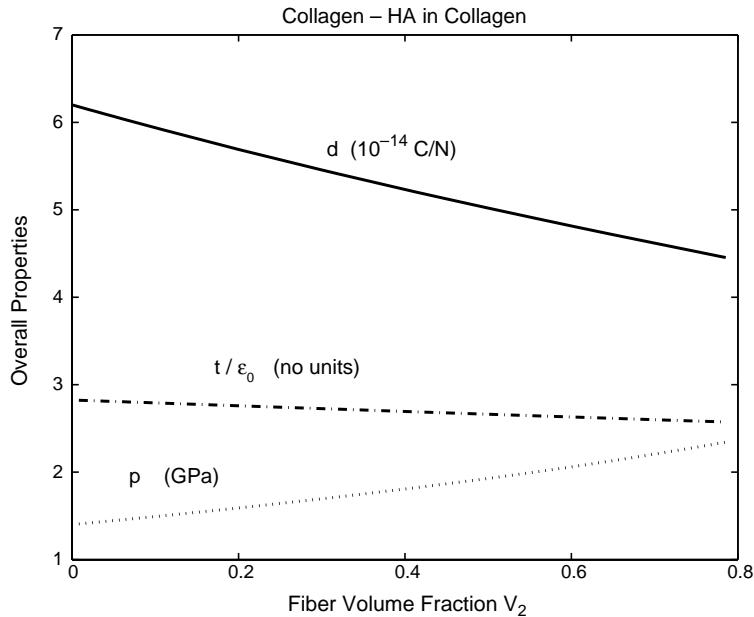


Fig. 2. Overall properties d , in units of 10^{-14} C/N, t/ϵ_0 (no units) and p (in GPa) plotted against fiber volume fraction V_2 . Binary fiber-reinforced composite: matrix of collagen with embedded fibers of a hydroxyapatite (HA) in collagen composite.

$d_2 = 0.041$ pC/N, where d_1 , d_2 are the shear strain piezoelectric coefficients. They are related to the shear stress piezoelectric ones used in the formulation above through the equation

$$s' = dp \quad (5.1)$$

(Ikeda, 1990, p. 17).

The infinite vectors and matrix in (4.26) and (4.27), which give the coefficients a_1 and b_1 of the overall properties in (4.10a,b), are truncated to a finite order, which is not a very large number to achieve enough accuracy. The results of the calculation are displayed in Fig. 2, which shows the overall properties d , t/ϵ_0 and p as a function of the fiber volume (area) fraction V_2 up to the percolation limit when cylinders get in contact. Each property shows a simple monotonic behaviour with V_2 .

6. Concluding remarks

The explicit formulae that were obtained for the overall properties p , s' , (4.10a,b) and its companion equation (4.26) is typical of the kind of results that can be derived using the methodology of this paper. The formulae show the dependence on the properties of the phases through Φ and the coefficients in (4.10a,b). The radius R of the cylinder, which is a number not greater than 1/2, appears in $B = (R \ 0)^T$ and w_{lk} , being proportional to R^{k+l} , is a very small number for large $k+l$. The terms $w_{11} = \pi R^2 = -w'_{11}$ in Ψ are interesting because the factor π that arises there is the necessary condition of doubly periodicity of the displacement $M^{(1)}$ and potential $N^{(1)}$. It does not involve the calculation of the particular lattice sum S_2 , that requires summation over a “needle”-shaped region, in similar problems which involve transport properties of regular arrays of cylinders using Rayleigh's method (1892) (Perrins et al., 1979). Note that $w_{11} = V_2$ is the volume fraction occupied by the fiber. The square array induces its particular geometric fea-

ture by means of the lattice sums which appear in w_{lk} besides the early consideration of summations over odd indices in (4.1a–d) (the square symmetry).

The formulae that were found for the effective properties may be useful as a benchmark to check numerical codes and experimental data.

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Appendix A. Overall properties p , s' , t

From the general formulae in (II, 2.12), it follows that

$$\begin{aligned}
 p &= p_v + \langle p_{13}M_{,1} - s'_{13}N_{,2} \rangle, \\
 &= p_v + \langle p_{23}M_{,2} - s'_{23}N_{,1} \rangle, \\
 s' &= s'_v + \langle s'_{13}M_{,1} + t_{13}N_{,2} \rangle, \\
 &= s'_v + \langle s'_{23}M_{,2} + t_{23}N_{,1} \rangle, \\
 &= s'_v + \langle p_{1}P_{,2} + s'_{1}Q_{,1} \rangle, \\
 &= s'_v + \langle p_{2}P_{,1} + s'_{2}Q_{,2} \rangle, \\
 t &= t_v + \langle -s'_{1}P_{,2} + t_{1}Q_{,1} \rangle, \\
 &= t_v + \langle s'_{2}P_{,1} + t_{2}Q_{,2} \rangle,
 \end{aligned} \tag{A.1a–h}$$

where

$$\begin{aligned}
 p_v &= V_1p_1 + V_2p_2, \\
 s'_v &= V_1s'_1 + V_2s'_2, \\
 t_v &= V_1t_1 + V_2t_2, \\
 V_1 + V_2 &= 1, V_2 = \pi R^2, \\
 \langle F(\mathbf{y}) \rangle &= \int_S F(\mathbf{y}) d\mathbf{y}.
 \end{aligned} \tag{A.2}$$

Table 1 describes the functions which appear in (A.1a–h)

Appendix B

$$\begin{aligned}
 p &= p_v - \|p_{\mathcal{Y}}\| \int_{\Gamma} 13M^{(2)} dy_2 - \|s'_{\mathcal{Y}}\| \int_{\Gamma} 13N^{(2)} dy_1, \\
 &= p_v + \|p_{\mathcal{Y}}\| \int_{\Gamma} 23M^{(2)} dy_1 + \|s'_{\mathcal{Y}}\| \int_{\Gamma} 23N^{(2)} dy_2,
 \end{aligned}$$

$$\begin{aligned}
s' &= s'_v - \|s'_T\| \int_{\Gamma} {}_{13}M^{(2)} dy_2 + \|t_T\| \int_{\Gamma} {}_{13}N^{(2)} dy_1, \\
&= s'_v + \|s'_T\| \int_{\Gamma} {}_{23}M^{(2)} dy_1 - \|t_T\| \int_{\Gamma} {}_{23}N^{(2)} dy_2, \\
&= s'_v + \|p_T\| \int_{\Gamma} {}_1P^{(2)} dy_1 + \|s'_T\| \int_{\Gamma} {}_1Q^{(2)} dy_2, \\
&= s'_v - \|p_T\| \int_{\Gamma} {}_2P^{(2)} dy_2 + \|s'_T\| \int_{\Gamma} {}_2Q^{(2)} dy_1, \\
t &= t_v - \|s'_T\| \int_{\Gamma} {}_1P^{(2)} dy_1 + \|t_T\| \int_{\Gamma} {}_1Q^{(2)} dy_2, \\
&= t_v - \|s'_T\| \int_{\Gamma} {}_2P^{(2)} dy_2 + \|t_T\| \int_{\Gamma} {}_2Q^{(2)} dy_1.
\end{aligned} \tag{B.1}$$

Final formulae for effective coefficients, in terms of the residue of solutions for each local problem

$$\begin{aligned}
p &= p_1(1 - 2\pi {}_{13}a_1), \\
&= p_1(1 + 2\pi {}_{23}a_1), \\
s' &= s'_1 + 2\pi t_1 {}_{13}b_1, \\
&= s'_1 + 2\pi t_1 {}_{23}b_1, \\
&= s'_1 + 2\pi p_1 {}_1a_1, \\
&= s'_1 + 2\pi p_1 {}_2a_1, \\
t &= t_1(1 - 2\pi {}_1b_1), \\
&= t_1(1 + 2\pi {}_2b_1).
\end{aligned} \tag{B.2a–h}$$

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